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LETTER TO THE EDITOR

Suppression of dissipation by localization

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Abstract. When expressed in an adiabatic basis, the evolution operator of a generic system with a time-dependent Hamiltonian is represented by a banded matrix, and the eigenstates of the evolution operator are localized. If the Hamiltonian is periodic in time, this result implies that the system can only absorb energy for a limited number of cycles. One consequence of this result is that, ideally, the absorption of low-frequency electromagnetic radiation by small, irregularly shaped, conducting particles can exhibit a non-classical saturation. The conditions under which this effect might be observable are discussed.

In this letter we consider the evolution operator $\hat{U}(t)$ for a system with a Hamiltonian which is periodic in time, and relate the results to a model for dissipation in microscopic quantum systems. We consider systems where the Hamiltonian is either random, or where the statistics of the spectrum can be modelled by random matrix theory and in which the variation of the Hamiltonian is slow compared to any relevant classical timescale. Given these conditions our results have a universal validity. We consider the application of our results to a system of 'independent electrons' moving inside an irregularly shaped enclosure, being subjected to a uniform electric field which varies periodically in time, with a frequency which is small compared to that of collisions with the walls. This is a realistic model for the absorption of low-frequency monochromatic electromagnetic radiation by small conducting particles. We will show that the eigenstates of the evolution operator are localized, and that this leads to a potentially observable suppression of the absorption of radiation by the particles. The results are related to other recent work involving the suppression of energy transfer due to localization of the eigenvectors of unitary operators, including studies on the quantized standard map (Grempel *et al* 1982) and a model for the response of microscopic rings to a time dependent flux (Gefen and Thouless 1987). Our results have a very wide range of potential applications because they refer to generic systems.

The system is defined by a single electron Hamiltonian $\hat{H}$ which depends on a parameter $X(t)$, which oscillates sinusoidally with frequency ω and amplitude X_0: $X(t) = X_0 \cos(\omega t)$. The response of the system to the slow perturbation is characterized by the following parameters: X_0, ω, $\hbar$, the density of states at the Fermi energy ρ_F, and the typical size σ_F of the off-diagonal matrix elements $\langle n|\partial\hat{H}/\partial X|m\rangle$ for states near the Fermi energy (here $|n\rangle$, $|m\rangle$ are instantaneous eigenstates of $\hat{H}$, and we define σ_F^2 to be the variance of the matrix elements with $E_n \simeq E_m \simeq E_F$). From these five parameters we can form two independent dimensionless groups, $\varkappa = \rho_F^2 \hbar \sigma_F X_0 \omega$ and $\chi = \rho_F \sigma_F X_0$: $\varkappa$ characterizes the adiabaticity of the motion (the quantum adiabatic theorem (Bohm 1951) applies if $\varkappa \ll 1$), and χ characterizes whether it is perturbative (energy levels are not mixed if $\chi \ll 1$).

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We consider the evolution operator $\hat{U}(t)$ in an adiabatic basis, i.e. in the basis formed by the eigenstates $|n(t)\rangle$ of the instantaneous Hamiltonian $\hat{H}(X(t))$. If the variation of $\hat{H}$ is sufficiently slow, the adiabatic theorem shows that the matrix representing $\hat{U}(t)$ is diagonal. For faster variations of the Hamiltonian the matrix is not diagonal but banded in the sense that the matrix elements decay rapidly away from the diagonal. The evolution operator can be characterized by a function $P(n)$, which represents the probability of making a transition from the ith state to the $(i+n)$th state:

$$P(n) = \langle |U_{i,i+n}|^2 \rangle \tag{1}$$

where $\langle\ \rangle$ denotes an average over states i near the Fermi energy. The form of the function $P(n)$ depends on the dimensionless parameters $\varkappa$ and χ, and several regimes can be distinguished, which will be discussed fully in a later publication. Here we examine only two of these regimes: we consider the limiting cases $\chi \to 0$ and $\chi \to \infty$, with $\varkappa \gg 1$ held fixed. In the limit $\chi \to 0$, the evolution operator can be calculated using perturbation theory, and it is easy to show that, after $N \gg 1$ cycles,

$$P(n) = (1-2p)\delta(n) + p\delta(n - \rho_F \hbar\omega) + p\delta(n + \rho_F \hbar\omega) \tag{2}$$

where $p = \pi^2 N\chi^3/\varkappa$. This result remains valid provided N is small enough that $p \ll 1$. In the limit $\chi \to \infty$ on the other hand, it is possible to show that $P(n)$ is essentially Gaussian, with a variance

$$\Delta_F = 2\pi^2 \varkappa\chi. \tag{3}$$

When $\chi \gg 1$, $P(n)$ deviates from a Gaussian in two ways. Firstly, the diagonal elements have twice the variance of the nearby off-diagonal elements if the system has time reversal invariance (Altshuler and Aronov 1981). Secondly, the decay is not as rapid as a Gaussian for very large $|n|$, but it can be shown that it is at least as rapid as $|n|^{-4}$, which is sufficient to guarantee the localization properties which are required.

It is well known that the eigenvectors of a banded Hermitian matrix with random elements are localized (see, e.g. Yeung and Oono (1987) and references therein) and the arguments leading to this result are equally valid for banded, random, unitary operators. Later we will present a model in which the evolution operator has been calculated and shown to have the properties described above, and where the eigenvectors of the evolution operator have localization properties similar to those of random unitary matrices with the same distribution $P(n)$.

This localization of the eigenstates of the evolution operator has important implications for the ability of the periodically perturbed system to exchange energy with the driving system. We illustrate this point by considering our model for the absorption of radiation by small conducting particles. Assume that the system starts in its ground state, with all of the energy levels up to N_F filled. After a time t, the total energy of the system of electrons is

$$E_T = \sum_{i=1}^{N_F} \sum_j |U_{ji}|^2 E_j \tag{4}$$

where the E_j are the energies of the single electron states and the U_{ij} are matrix elements of the evolution operator. The energy absorbed by the system is related to the distribution $P(n)$: an elementary calculation shows that the energy absorbed is

$$\Delta E_T = \Delta_F / 2\rho_F \qquad \Delta_F = \int_{-\infty}^{\infty} dn\, n^2 P(n) \tag{5}$$

where Δ_F is the second moment of the distribution $P(n)$. We can compare this with the conventional way of calculating the energy dissipated using the Kubo-Greenwood formula (Kubo 1956, Greenwood 1957), which can be expressed in the form (Wilkinson 1988, 1990)

$$\frac{dE_T}{dt} = \pi\rho_F^2\sigma_F^2\hbar\left(\frac{dX}{dt}\right)^2. \tag{6}$$

It is easy to verify that the estimate for ΔE_T obtained by integrating this formula is the same as that obtained from (5), using the properties of the distribution function $P(n)$ described above.

For a periodically driven system the amount of energy absorbed after the Nth cycle will be determined by the value of Δ_F for the Nth power of the Floquet operator (the evolution operator for one cycle). If the eigenstates of the Floquet operator are localized, with a localization length λ, the value of Δ_F cannot exceed a maximum value $\Delta_{max} = O(\lambda^2)$ and after initially increasing linearly at a rate given by the Kubo-Greenwood formula the energy absorbed will saturate. When $\chi \gg 1$, numerical experiments and theoretical arguments show that $\lambda = O(\Delta)$, implying that the energy absorption will saturate after $O(N^*)$ cycles, where $N^* = O(\Delta) = O(\varkappa\chi)$. When $\chi \ll 1$, the localization length cannot be greater than $O(\rho_F\hbar\omega) = O(\varkappa/\chi)$, so that $N^* = O(\varkappa/\chi^3)$. Note that these results imply that the saturation effect is most pronounced (i.e. N^* is small) when χ is neither very large nor very small.

These conclusions are surprising because they imply that a reasonable model for small metallic particles exhibits a non-classical saturation of the absorption of low-frequency monochromatic radiation: we must consider carefully the conditions required to observe this effect. The result depends on the Hamiltonian being a periodic function of time, and we argue that the effect appears to be extremely sensitive to deviations of the Hamiltonian from exact periodicity. In the case of the application to the absorption of radiation, the deviations from periodicity can be caused by the random perturbation of the small particles by phonons in the supporting material. If $|u_i^N\rangle$ is the ith eigenvector of the evolution operator at the Nth cycle of the motion, the condition for the suppression of dissipation is clearly that the matrix of overlaps of eigenvectors for successive cycles be close to the identity:

$$\langle u_i^N | u_j^{N+1}\rangle = \delta_{ij} + O(1/\sqrt{N^*}) \tag{7}$$

where N^* is the number of cycles after which saturation is expected to occur. In order to satisfy this criterion the non-periodic perturbation of the Hamiltonian must satisfy two conditions. If the Hamiltonian were time independent, the eigenstates of $\hat{U}$ would be the same as those of $\hat{H}$, and the condition (7) would be the condition for states not to be mixed by the perturbation, i.e. the shift δE in the energy levels due to the non-periodic perturbation should satisfy

$$\rho_F\delta E \ll 1/\sqrt{N^*}. \tag{8}$$

Also, in order to maintain the same phase relationships in the evolution operator, δE should satisfy

$$\frac{1}{\hbar}\int_0^T dt\,\delta E(t) \ll 1/\sqrt{N^*} \tag{9}$$

where $T = 2\pi/\omega$ is a period of the perturbation. The conditions (8) and (9) represent a severe limitation on the size of the non-periodic perturbation δE.

Now we discuss the numerical results which support the claim that the energy absorption saturates. We used a time-dependent extension of the well known Gaussian orthogonal ensemble (GOE) random matrix model (Porter 1965): this model is appropriate for describing a disordered system in a small range of energies about the Fermi level, in situations where (spatial) localization does not play a rôle (Gorkov and Eliashberg 1965, Altshuler and Shklovskii 1986). The GOE model is a symmetric $N \times N$ random matrix, with independently Gaussian distributed elements, which have mean value zero and variance $\sigma^2(1+\delta_{ij})$. The Fermi energy was placed at the centre of the spectrum, where the density of states is $\rho = \sqrt{N}/\sigma\pi$. The matrix elements $\langle n|\partial H/\partial X|m\rangle$ (where the perturbation parameter X represents the electric field in our example) can also be modelled by a GOE matrix, provided ω^{-1} is larger than the classical characteristic timescale associated with the electron motion (for a discussion of this point see Gorkov and Eliashberg (1965)). Time dependence was therefore introduced by taking the Hamiltonian to be a linear combination of two GOE matrices, $\hat{H}_1$ and $\hat{H}_2$:

$$\hat{H} = \cos X\,\hat{H}_1 + \sin X\,\hat{H}_2 \qquad X(t) = X_0 \cos(\omega t). \tag{10}$$

This model is a GOE with the same statistical properties for all X. We evaluated the evolution operator for this model by integrating the Schrödinger equation using the fourth-order Runge-Kutta method, and transforming the result to an adiabatic basis. The results discussed below are for the following parameter values: $\sigma = \hbar = 1$, $N = 100$, $X_0 = \pi/\sqrt{N}$, $\omega = 2X_0$, so that the dimensionless parameters are $\chi = 1$, $\varkappa = 2$ (this required about 25 minutes on a Cray computer for each realization).

Figure 1 is a plot of the distribution function $P(n)$, plotted on a logarithmic scale. Despite the fact that $\chi = 1$ is not large, the results are already in good agreement with the theoretical predictions for the $\chi \to \infty$ limit: the curve is a good fit to a Gaussian, with variance $\Delta_F = 41.6$, close to the prediction given by (3). The non-Gaussian tail of the distribution vanishes more rapidly than $|n|^{-4}$. Figure 2 (curve A) shows the saturation

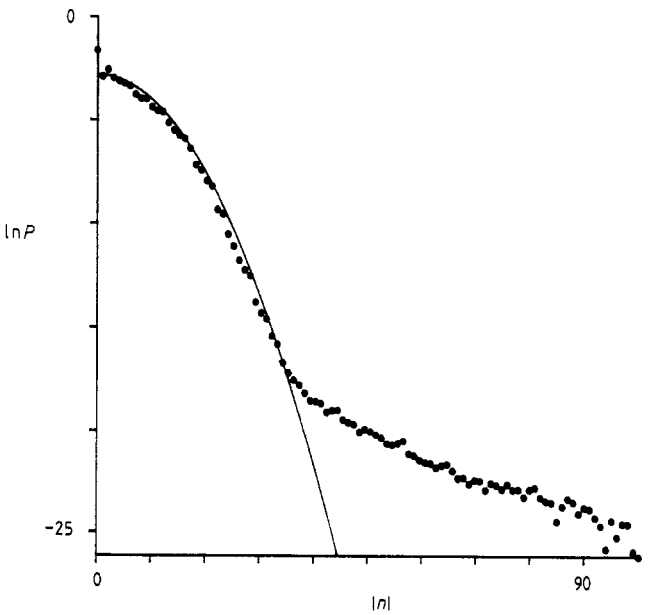

Figure 1. Logarithmic plot of the distribution function $P(n)$ for the evolution operator of the GOE model, showing a fit to a quadratic. The non-Gaussian tail vanishes faster than $|n|^{-4}$.

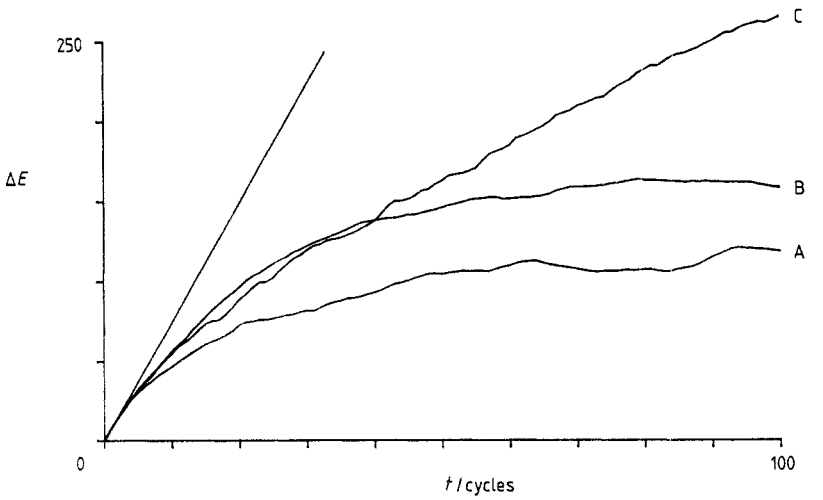

Figure 2. Curve A: energy absorbed as a function of time for the random matrix model (9); the energy initially increases at a rate given by the Kubo formula (straight line), but soon starts to saturate. Curve B is a simulation using a random unitary matrix, and curve C illustrates the effect of introducing noise.

of the energy absorption calculated using (4): initially the energy increases linearly, at a rate predicted by the Kubo formula (straight line on figure 2). Curve B is the result obtained using a random, Gaussian-banded unitary matrix with the same value of Δ_F as the evolution operator: our numerical results are consistent with the localization properties of the evolution operator being statistically equivalent to those of these random matrices. Curve C is an illustration of the sensitivity of the localization effect to noise: we evaluated a second evolution operator, for the same realization of the GOE, but with the frequency ω increased by 5%, and evolved the wavefunction by multiplying a random sequence of the two operators (each occurring with probability $\frac{1}{2}$). The localization effect is suppressed in this case, and the energy absorbed does not saturate until all the states are filled with equal probability.

We also investigated the localization length λ (which we defined as the square root of the second moment of the probability distribution of an eigenstate) systematically in the regime where $P(n)$ is approximately Gaussian, using Gaussian-banded random unitary matrices to simulate evolution operators when calculation of the latter would have been prohibitively slow. We found that λ is proportional to Δ_F, and that the multiplier is approximately unity.

Having discussed the saturation effect in general terms we now describe some estimates which are required for applying the predictions to the absorption of radiation by small conducting particles, where the electric field $\mathscr{E}(t)$ of the applied radiation plays the rôle of the time-dependent parameter $X(t)$. Estimating σ_F, the typical size of the dipole matrix element, is not entirely straightforward because the applied electric field is screened by the electrons (this point was not treated properly by Gorkov and Eliashberg: see Strässler *et al* 1972). The easiest method is to use the fact that, when $\varkappa \gg 1$, the initial rate of dissipation is given correctly by the Kubo-Greenwood theory (6), and to equate this to a classical estimate for the rate of dissipation. Applying an external electric field $\mathscr{E}$ causes a charge q to appear on the surface of the particle which screens out the applied field: $q \simeq \mathscr{E}\varepsilon_0\varepsilon_r a^2$, where a is the typical dimension of

the particles. This transfer of charge implies that a current $I \simeq q\omega$ flows through the particle, with a consequent dissipation of energy at a rate RI^2, where R is the effective resistance of the particle. We can use the Drude model to estimate the resistance, replacing the bulk relaxation time by a/v_F if the particle is smaller than the bulk mean free path, where v_F is the Fermi velocity. This leads to a semiclassical estimate for the rate of dissipation which is equated to (6) to give an estimate for σ_F. Absorption by eddy currents induced by the magnetic field of the radiation may be the dominant effect in very small systems (see Carr *et al* (1985) for a review of optical properties of small metal particles); these can be treated in a similar way to the electric field effect considered above.

We estimate the size of the perturbation due to phonons using the following simple model. We regard the small metallic particles as conducting regions of typical size a embedded in a non-conducting continuum, with bulk modulus K, speed of sound c and temperature T. The perturbation δE due to the phonons is given by $\delta E \simeq \delta a(E_F/a)$ where δa is the RMS change in the dimension of the particle due to phonons. The RMS displacement of a point due to a phonon with wavevector k is $\delta x_k^2 = O(k_B T/KVk^2)$, where V is the volume of the sample, provided the energy of the phonon is small enough for it to be excited thermally, i.e. $k < k_{max} = k_B T/\hbar c$. The relative displacement of the two sides of a particle of size a due to this phonon is $\delta a_k \simeq \delta x_k$, if $ka \gg 1$, otherwise $\delta a_k \simeq ka\delta x_k$ if $ka \ll 1$. Combining these δa_k incoherently for all phonons with $k < k_{max}$, assuming that most of these phonons have $ka \gg 1$, gives the estimate

$$\delta a \simeq \left(\frac{k_B T}{\hbar c}\right)(\hbar c/K)^{1/2}. \tag{11}$$

We mentioned earlier that the saturation effect will be most readily observable in systems in which χ is of order unity and $\varkappa$ is not too large. These conditions are difficult to satisfy for three-dimensional particles at frequencies large enough for the absorption of radiation to be experimentally observable. If the small conducting particles are prepared by etching a surface holding a two-dimensional electron gas, it appears to be possible to reach the required experimental conditions. Assuming a Fermi energy of 10 meV, an effective mass $m^* = 0.1\ m_e$, relative permittivity $\varepsilon_r = 10$, and that the electron motion is ballistic for the particle sizes we consider, we estimate $\rho_F/a^2 = 1.5 \times 10^{36}\ J^{-1}\ m^{-2}$, and $\sigma_F^2 a = 4 \times 10^{-63}\ C^2\ m^3$ (we are ignoring spin and other degeneracies). For particles of size $a \simeq 10^{-6}$ m, we have $\chi = 1$, $\varkappa = 2$ (the parameters of the numerical simulation above) at a frequency of around 20 GHz, and field of $10^4\ V\ m^{-1}$. We conclude that the parameter regimes where the saturation effect is most readily observed are attainable in two-dimensional systems. We estimated the RMS strain on these particles due to phonons, using the model described above, assuming $c = 10^4\ m\ s^{-1}$, $K = 10^{11}\ N\ m^{-2}$, and $T = 10$ K: we find $\delta E/E \simeq 10^{-7}$, which implies that the inequalities (8) and (9) are easily satisfied, so that the effect need not be destroyed by phonons.

In conclusion, we have shown that localization of the eigenstates of the evolution operator leads to a suppression of absorption of energy from a periodic disturbance in generic quantum systems. The effect is very sensitive to noise. Simple estimates suggest that the effect could be observable in absorption of microwave radiation by two-dimensional particles.

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